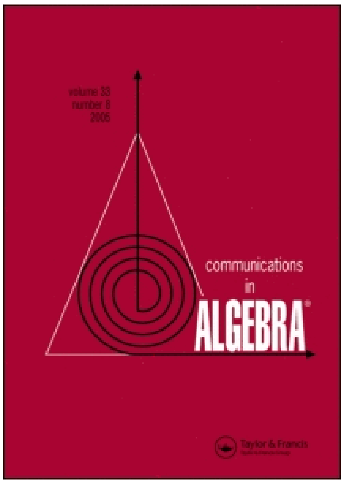

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On the γ-Cyclic Hypergroups

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ON THE γ-CYCLIC HYPERGROUPS

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The class of γ-complete hypergroups and γ-cyclic hypergroups is introduced. Several properties and examples are found.

Key Words: γ-closure; γ-complete hypergroup; γ-cycle hypergroup.

AMS Mathematics Subject Classification: 20N20.

1. INTRODUCTION

In this article we use the definition of hypergroup introduced by Marty (1934). Let H be a hypergroup and $\mathcal{U}$ be the set of all finite products of elements of H. The relation β is defined on H as follows:

$$x\beta y \text{ if and only if } \{x, y\} \subseteq u \qquad \text{for some } u \in \mathcal{U}.$$

The relation β was introduced on hypergroups by Koskas (1970) and was studied mainly by Corsini (1993) and Vougiouklis (1994). The fundamental relation β^* is the smallest equivalence relation on H such that H/β^*, the set of all equivalence classes, is a group. Freni (1991) proved that in hypergroups, the relation β is transitive and $\beta^* = \beta$. Using the relation β, Migliorato (1994) defined the notion of *complete hypergroups*. Cyclic hypergroups already considered at the begining of the theory's history (Wall, 1937) have been later on studied in depth by Vougiouklis (1981) and afterwards by Konguetsof et al. (1986) and Leoreanu (2000). Cyclic hypergroups are important not only in the sphere of finitely generated hypergroups but also for interesting combinatorial implications. Let $\phi : H \longrightarrow H/\beta^*$ be the *canonical projection*. A hypergroup H is called *cyclic* with a *generator* x if $\phi(H)$ is a cyclic group generated by $\phi(x)$.

Recently, Freni (2002) introduced the relation γ as a generalization of the relation β. If H is a hypergroup, then γ^* denotes the *transitive closure* of the relation $\gamma = \bigcup_{n\geq 1} \gamma_n$, where γ_1 is the diagonal relation and for every integer $n > 1$, γ_n is the

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relation defined as follows:

$$x\gamma_n y \Longleftrightarrow \exists(z_1, z_2, \ldots, z_n) \in H^n, \qquad \exists\sigma \in \mathbb{S}_n : x \in \prod_{i=1}^{n} z_i,\ y \in \prod_{i=1}^{n} z_{\sigma(i)},$$

where $\mathbb{S}_n$ is the symmetric group of order n. Freni proved that γ^* is the smallest strongly regular equivalence relation (cf. Freni, 2002, Theorem 1.1) such that H/γ^* is an Abelian group (cf. Freni, 2002, Corollary 1.2). Also, he determined some necessary and sufficient conditions so that the relation γ is transitive (cf. Freni, 2002, Theorem 2.3).

Using the relation γ we shall introduce in section 3 the γ-closure of a subset A and we study the properties of $C_\gamma(A)$, where $C_\gamma(A)$ is the intersection of all γ-parts containing A. In Section 4, we introduce the γ-complete semi-hypergroups. A semi-hypergroup H is *γ-complete* if satisfies one of the equivalent conditions:

(1) $\forall(x, y) \in H^2, \forall\sigma \in \mathbb{S}_2, \forall a \in x_{\sigma(1)} \circ y_{\sigma(2)} : C_\gamma(a) = x_{\sigma(1)} \circ y_{\sigma(2)}$;
(2) $\forall(x, y) \in H^2, \forall\sigma \in \mathbb{S}_2 : C_\gamma(x_{\sigma(1)} \circ y_{\sigma(2)}) = x_{\sigma(1)} \circ y_{\sigma(2)}$;
(3) $\forall(m, n) \in \mathbb{N}^2, 2 \le m, n, \forall(x_1, \ldots, x_n) \in H^n, \forall(y_1, \ldots, y_m) \in H^m, \forall(\sigma, \tau) \in \mathbb{S}_n \times \mathbb{S}_m$:

$$\prod_{i=1}^{n} x_{\sigma(i)} \cap \prod_{j=1}^{m} y_{\tau(j)} \neq \emptyset \Longrightarrow \prod_{i=1}^{n} x_{\sigma(i)} = \prod_{j=1}^{m} y_{\tau(j)}.$$

We shall introduce the γ-cyclic hypergroup in Section 5. A hypergroup H is a *γ-cyclic hypergroup with a generator x* if H/γ^* is a cyclic group generated by $\phi_H(x)$, where $\phi_H : H \to H/\gamma^*$ is the canonical projection. In Section 5, we show that every γ-cyclic and γ-complete hypergroup is commutative.

2. NOTATIONS AND PRELIMINARIES

A *semi-hypergroup* $(H, \circ)$ is a nonempty set H equipped with a hyperoperation $\circ$, that is a map $\circ : H \times H \longrightarrow \wp^*(H)$, where $\wp^*(H)$ denotes the family of all nonempty subsets of H, and for all $(x, y, z) \in H^3 : x \circ (y \circ z) = (x \circ y) \circ z$. A semi-hypergroup H is said to be a *hypergroup* if for every $a \in H : a \circ H = H \circ a = H$. In the above definitions, if $A, B \in \wp^*(H)$, then we mean $A \odot B$ by

$$A \odot B = \bigcup_{a\in A, b\in B} a \circ b.$$

Let $(H, \circ)$ be a hypergroup and $R \subseteq H \times H$ an equivalence relation. If $\{A, B\} \subseteq \wp^*(H)$ then we set:

$$A\,\overline{\overline{R}}\,B \Longleftrightarrow a\,R\,b, \qquad \forall a \in A, \quad \forall b \in B.$$

If $(H, \circ)$ is a semi-hypergroup, the relation R is said to be *strongly right-regular* (resp. *left-regular*) if

$$x\,R\,y \Longrightarrow x \circ a\overline{\overline{R}}\,y \circ a$$

$$(\text{resp. } x\,R\,y \Longrightarrow a \circ x\overline{\overline{R}}\,a \circ y),$$

for all $(x, y, a) \in H^3$. Moreover, R is called *strongly regular* if it is strongly regular to the right and to the left.

For every $n \in \mathbb{N}$, we shall write $(\beta_n^*)_H$ to denote the transitive closure of the relation $(\beta_n)_H$ define as follows: $\forall (x, y) \in H^2, x(\beta_n)_H y \Longleftrightarrow \exists (z_1, z_2, \ldots, z_n) \in H^n$ such that $\{x, y\} \subseteq \prod_{i=1}^n z_i$. Moreover, one puts $(\beta_1)_H = \{(x, x) \mid x \in H\}$ and $(\beta)_H = \bigcup_{n\in\mathbb{N}}(\beta_n)_H$. We always use α^* to show the transitive closure of α and it is known that in every hypergroup $\beta_H = \beta_H^*$, see Freni (2002). When it is understood which hypergroup is being considered β, β_n, and β_n^* will be written in place of $(\beta)_H$, $(\beta_n)_H$, and $(\beta_n^*)_H$, respectively. In general, if R is an equivalence relation on a set A, then for every $S \in \wp^*(A)$, we shall put $R(S) = \bigcup_{x\in S} R(x)$.

Given a hypergroup H, a quotient H/R of H modulo an regular equivalence relation R becomes a hypergroup under the following hyperoperation:

$$\forall (\overline{x}, \overline{y}) \in (H/R)^2, \qquad \overline{x} \otimes \overline{y} = \{\overline{z} \mid z \in x \circ y\}.$$

If R is a strongly regular equivalence on H, then H/R is a group (cf. Corsini, 1993, Theorem 31). We recall that β is the smallest strongly regular equivalence on H (cf. Corsini, 1993, Theorem 12). Let $\phi : H \longrightarrow H/\beta$ be the canonical projection, the *heart* of H is the set $w_H = \phi^{-1}(1_{H/\beta})$. One of the most important notions in hypergroup theory is the heart of a hypergroup. The knowledge of this concept gives information on the structure of the hypergroup H and in some cases determines completely.

Complete parts, introduced and studied for the first time by Koskas (1970) were subsequently analyzed by Corsini (1993). Let A be a part of a semi-hypergroup H, it means A is subset of H, A is called *complete* if the following implication is valid:

$$\forall n \in \mathbb{N}, \quad \forall (x_1, \ldots, x_n) \in H^n, \qquad \prod_{i=1}^n x_i \cap A \neq \emptyset \Longrightarrow \prod_{i=1}^n x_i \subseteq A.$$

Let A be a nonempty part of H. The intersection of the parts of H which are complete and contain A is called the *complete closure* of A in H; it will be denoted by $C(A)$. A semi-hypergroup H is complete (Migliorato, 1994), if it satisfies one of the following conditions:

(1) $\forall (x, y) \in H^2, \forall a \in x \circ y, C(a) = x \circ y$;
(2) $\forall (x, y) \in H^2, C(x \circ y) = x \circ y$;
(3) $\forall (m, n) \in \mathbb{N}^2, 2 \leq m, n, \forall (x_1, \ldots, x_n) \in H^n, \forall (y_1, \ldots, y_m) \in H^m$,

$$\prod_{i=1}^n x_i \cap \prod_{j=1}^m y_j \neq \emptyset \Longrightarrow \prod_{i=1}^n x_i = \prod_{j=1}^m y_j.$$

A hypergroup H is called *cyclic hypergroup with generator* x (Wall, 1937) if $\phi(H)$ is a cyclic group generated by $\phi(x)$. An element y in a hypergroup H is called *periodical* of period n if $x^n \subseteq w_H$ and write $cycle(y) = n$, where w_H is the heart of hypergroup H and $n = \min\{k \in \mathbb{N} \mid x^k \subseteq w_H\}$. We will write $p(x) = n$. A semi-hypergroup H is called *cyclic* (Wall, 1937) if there exists $h \in H$ such that for every $x \in H$, there exists $n \in \mathbb{N}$ such that $x \in h^n$. We call h the *s-generator* of H. A hypergroup H is called *s-cyclic* if it is a cyclic semi-hypergroup. If H is a cyclic semi-hypergroup s-generated from h and $a \in H$, we call the cyclicly of the integer $m = \min\{q \in \mathbb{N}^* - \{1\} \mid a \in h^q\}$. We will write $cycle(a) = m$.

We recall the following definition from Freni (2002). If H is a semi-hypergroup, then we set: $\gamma_1 = \{(x, x) \mid x \in H\}$ and for every integer $n > 1$,

$$x\gamma_n y \Longleftrightarrow \exists(z_1, z_2, \ldots, z_n) \in H^n, \qquad \exists\sigma \in \mathbb{S}_n : x \in \prod_{i=1}^{n} z_i \text{ and } y \in \prod_{i=1}^{n} z_{\sigma(i)}.$$

Obviously, for every $n \geq 1$, the relation γ_n is symmetric and the relation $\gamma = \bigcup_{n\geq 1} \gamma_n$ is reflexive and symmetric. Let γ^* be the transitive closure of γ. The relation γ^* is a strongly regular equivalence relation (cf. Freni, 2002, Theorem 1.1), and if H is a hypergroup, then $\gamma = \gamma^*$ (cf. Freni, 2002 Theorem 3.3) and H/γ^* is an Abelian group (cf. Freni, 2002, Corollary 1.2).

Let M be a nonempty subset of a semi-hypergroup H, we say that M is a γ-part of H (Freni, 2002), if for every $n \in \mathbb{N}^*$, for every $(z_1, z_2, \ldots, z_n) \in H^n$ and for every $\sigma \in \mathbb{S}_n$, we have

$$\prod_{i=1}^{n} z_i \cap M \neq \emptyset \Longrightarrow \prod_{i=1}^{n} z_{\sigma(i)} \subseteq M$$

Let $\phi : H \to H/\gamma^*$ be the canonical projection. $D(H)$ is called *derived hypergroup* and we have $D(H) = \phi^{-1}(1_{H/\gamma^*})$ (cf. Freni, 2002, Theorem 3.1). We also have for every nonempty subset M of hypergroup H, $\phi^{-1}(\phi(M)) = D(H) \odot M = M \odot D(H)$ (cf. Freni, 2002, Theorem 3.2).

An element $e \in H$ is called an *identity* if $a \in e \circ a \cap a \circ e$ for all $a \in H$. An element x' is called an *inverse* of x if an identity e exists such that $e \in x \circ x' \cap x' \circ x$. An element u of a hypergroup H is called a *scalar identity* if $u \circ x = x \circ u =$ singleton set for all $x \in H$.

A hypergroup H is *regular* if it has at least one identity and every elements have at least one inverse. If H is regular, for every $a \in H$, we denote $i(x)$ the set of the inverses of x. A regular hypergroup is said to be *reversible*, if it satisfies the following conditions:

$$\forall(a, b, x) \in H^3 : a \in b \circ x \Longrightarrow \exists x' \in i(x) : b \in a \circ x',$$
$$a \in x \circ b \Longrightarrow \exists x'' \in i(x) : b \in x'' \circ a.$$

A commutative reversible hypergroup is called *canonical* if it has a scaler identity and every element has a unique inverse (cf. Corsini, 1993, Definition 165). A commutative reversible hypergroup is called *feebly canonical* (cf. Corsini, 1993, Definition 227) if

$$\forall(a, x) \in H^2, \quad \forall\{u, v\} \subseteq i(x), \qquad u \circ a = v \circ a. \tag{2.1}$$

3. γ-CLOSURE PARTS OF HYPERGROUPS

We begin with some properties of γ-parts of hypergroups which are valid in every hypergroups. We suppose that $H = (H, \circ)$ is a hypergroup and $\phi_H : H \longrightarrow H/\gamma^*$ is the canonical projection.

Definition 3.1. Let A be a nonempty subset of H. The intersection of γ-parts H which contain A is called *γ-closure* of A in H. It will be denoted $C_\gamma(A)$.

Theorem 3.2. *Let A be a nonempty subset of H. We pose*

$$G_1(A) := A,$$

$$G_{n+1}(A) := \left\{x \mid \exists p \in \mathbb{N}, \exists(h_1, \ldots, h_p) \in H^p, \exists \sigma \in \mathbb{S}_p : x \in \prod_{i=1}^{p} h_{\sigma(i)}, \prod_{i=1}^{p} h_i \cap G_n \neq \emptyset\right\},$$

$$G(A) := \bigcup_{1 \leq n} G_n(A).$$

Then $G(A) = C_\gamma(A)$.

Proof. It is necessary to prove:

(i) $G(A)$ is a γ-part of H;
(ii) If $A \subseteq B$ and B is a γ-part of H then $G(A) \subseteq B$.

Therefore,

(i) Let $\prod_{i=1}^{p} x_i \cap G(A) \neq \emptyset$ then there exists $n \in \mathbb{N}$ such that $\prod_{i=1}^{p} x_i \cap G_n(A) \neq \emptyset$. For every $\sigma \in \mathbb{S}_n$ and $y \in \prod_{i=1}^{n} x_{\sigma(i)}$ we have $y \in G_{n+1}(A)$ and $\prod_{i=1}^{n} x_{\sigma(i)} \subseteq G(A)$, and so $G(A)$ is a γ-part of H;
(ii) We have $A = G_1(A) \subseteq B$. Suppose that B is a γ-part of H and $G_n(A) \subseteq B$. We prove that this implies $G_{n+1}(A) \subseteq B$. For every $z \in G_{n+1}(A)$ there exist $p \in \mathbb{N}$, $(x_1, \ldots, x_p) \in H^p$ and $\sigma \in \mathbb{S}_p$ such that $z \in \prod_{i=1}^{p} x_{\sigma(i)}$, $\prod_{i=1}^{p} x_i \cap G_n(A) \neq \emptyset$. $G_n(A) \subseteq B$ thus $\prod_{i=1}^{p} x_i \cap B \neq \emptyset$ hence $z \in \prod_{i=1}^{p} x_{\sigma(i)} \subseteq B$ and so $G_{n+1}(A) \subseteq B$. □

Lemma 3.3. *We have*:

(i) $\forall n \geq 2, \forall x \in H, G_n(G_2(x)) = G_{n+1}(x)$;
(ii) $x \in G_n(y) \Leftrightarrow y \in G_n(x)$.

Proof. (i)

$$G_2(G_2(x)) = \left\{z \mid \exists p \in \mathbb{N}, \exists(h_1, \ldots, h_p) \in H^p, \exists \sigma \in \mathbb{S}_p : z \in \prod_{i=1}^{p} h_{\sigma(i)}, \prod_{i=1}^{p} h_i \cap G_2 \neq \emptyset\right\} = G_3.$$

We now shall proceed by induction: Suppose that $G_{n-1}(G_2(x)) = G_n(x)$, then

$$G_n(G_2) : = \left\{z \mid \exists p \in \mathbb{N}, \exists(h_1, \ldots, h_p) \in H^p, \exists \sigma \in \mathbb{S}_p : z \in \prod_{i=1}^{p} h_{\sigma(i)}, \prod_{i=1}^{p} h_i \cap G_{n-1}(G_2(x)) \neq \emptyset\right\}$$

$$= \left\{z \mid \exists p \in \mathbb{N}, \exists(h_1, \ldots, h_p) \in H^p, \exists \sigma \in \mathbb{S}_p : z \in \prod_{i=1}^{p} h_{\sigma(i)}, \prod_{i=1}^{p} h_i \cap G_n \neq \emptyset\right\}$$

$$= G_{n+1}(x).$$

(ii) We prove the affirmation by induction. It is clear that $x \in G_2(y) \Leftrightarrow y \in G_2(x)$. Suppose that $x \in G_{n-1}(y) \Leftrightarrow y \in G_{n-1}(x)$. Let $x \in G_n(y)$, then there exist $q \in \mathbb{N}$, $(a_1, \ldots, a_q) \in H^q$ and $\sigma \in \mathbb{S}_q$ such that

$$x \in \prod_{i=1}^{q} a_{\sigma(i)}, \prod_{i=1}^{n} a_i \cap G_{n-1}(y) \neq \emptyset,$$

and thus there exists $v \in \prod_{i=1}^{n} a_i \cap G_{n-1}(y)$. Therefore, by choosing $\sigma = 1$ the identity map, $v \in G_2(x)$ is obtained. From $v \in G_{n-1}(y)$ we have $y \in G_{n-1}(G_2(x)) = G_n(x)$. □

Theorem 3.4. *The relation $xGy \Leftrightarrow x \in G(\{y\})$ is an equivalence.*

Proof. We write $C_\gamma(x)$ instead of $C_\gamma(\{x\})$. G is clearly reflexive. Now, let xGy and yGz. If P is a γ-part of H and $z \in P$, then $C_\gamma(z) \subseteq P$, $y \in P$ and consequently $x \in C_\gamma(y) \subseteq P$. For this reason $x \in C_\gamma(z)$ that is xGz. The symmetrically of G follows in a direct way from the preceding lemma. □

Theorem 3.5. $\forall (x, y) \in H^2$, *one gets $xGy \Leftrightarrow x\gamma^* y$.*

Proof. Let $x\gamma y$ thus:

$$\exists n \in \mathbb{N} : x\gamma_n y \Longrightarrow \exists (z_1, \ldots, z_n) \in H^n, \exists \sigma \in \mathbb{S}_n : x \in \prod_{i=1}^{n} z_i,\ y \in \prod_{i=1}^{n} z_{\sigma(i)}.$$

We have $\prod_{i=1}^{n} z_i \cap \{x\} \neq \emptyset$, so

$$x \in G_2(y) \Longrightarrow x \in C_\gamma(y) \Longrightarrow xGy \Longrightarrow \gamma \subseteq G.$$

Since G is an equivalence relation, then $\gamma^* \subseteq G$.

Conversely, if xGy, then there exists $n \in \mathbb{N}$ such that $x \in G_{n+1}(y)$, from this it follows that $\exists m \in \mathbb{N}, \exists (z_1^1, \ldots, z_m^1) \in H^m, \exists \sigma^1 \in \mathbb{S}_n$:

$$x \in \prod_{i=1}^{m} z^1_{\sigma^1(i)}, \prod_{i=1}^{m} z_i^1 \cap G_n(y) \neq \emptyset,$$

thus $x_1 \in \prod_{i=1}^{m} z_i^1 \cap G_n(y)$. Therefore $x\gamma x_1$ and $x_1 \in G_n(y)$, and so there exist $r \in \mathbb{N}$, $(z_1^2, \ldots, z_r^2) \in H^r$, $\sigma^2 \in \mathbb{S}_r$ such that

$$x_1 \in \prod_{i=1}^{r} z^2_{\sigma^2(i)}, \prod_{i=1}^{r} z_i^2 \cap G_{n-1}(y) \neq \emptyset \Longrightarrow \exists x_2 \in \prod_{i=1}^{r} z_i^2 \cap G_{n-1}(y) \Longrightarrow x_1\gamma x_2.$$

So as a consequence, one obtains

$$\exists x_n \in \prod_{i=1}^{s} z^n_{\sigma^n(i)} \cap G_{n-(n-1)}(y) \Longrightarrow x_n \in G_1(y) = \{y\} \Longrightarrow x_n = y,$$

and so $x\gamma x_1 \ldots \gamma x_n = y$. Therefore $G \subseteq \gamma^*$. □

Theorem 3.6. *If B is a nonempty subset of H, one has $C_\gamma(B) = \bigcup_{b\in B} C_\gamma(b)$.*

Proof. It is clear for every $b \in B$, $C_\gamma(b) \subseteq C_\gamma(B)$, because every γ-part containing B contains $\{b\}$, therefore $\bigcup_{b\in B} C_\gamma(b) \subseteq C_\gamma(B)$. To prove the converse recall that $C_\gamma(B) = \bigcup_{1\leq n} G_n(B)$ by Theorem 3.2. We shall prove theorem by induction on n. Let $n = 1$ then by Theorem 3.2, we have

$$G_1(B) = B = \bigcup_{b\in B}\{b\} = \bigcup_{b\in B} G_1(b).$$

Suppose that the statement holds for n, that is, $G_n(B) \subseteq \bigcup_{b\in B} G_n(b)$ and we shall prove that $G_{n+1}(B) \subseteq \bigcup_{b\in B} G_{n+1}(b)$. If $z \in G_{n+1}(B)$, then there exist $q \in \mathbb{N}$, $(x_1, \ldots, x_q) \in H^q$, $\sigma \in S_q$ such that

$$z \in \prod_{i=1}^{q} x_{\sigma(i)}, \prod_{i=1}^{q} x_i \cap G_n(B) \neq \emptyset,$$

by induction $\prod_{i=1}^q x_i \cap (\bigcup_{b\in B} G_n(b)) \neq \emptyset$, hence there exists $b' \in B$ such that $\prod_{i=1}^q x_i \cap G_n(b') \neq \emptyset$. Since $z \in \prod_{i=1}^q x_{\sigma(i)}$ one gets $z \in G_{n+1}(b')$ and so one has proven $G_{n+1}(B) \subseteq \bigcup_{b\in B} G_{n+1}(b)$. Therefore $C_\gamma(B) \subseteq \bigcup_{b\in B} C_\gamma(b)$. □

Corollary 3.7. *If A is a γ-part of H, then $A \odot B$, $B \odot A$ are γ-parts of H for every $B \in \wp^*(H)$.*

Proof. We have $C_\gamma(A \odot B) = A \odot B \odot D(H) = A \odot D(H) \odot B = C_\gamma(A) \odot B = A \odot B$. □

Corollary 3.8. *Let $A \in \wp^*(H)$, then A is a γ-part of H if and only if $A \odot D(H) = A$.*

Proof. We have $C_\gamma(A) = A \odot D(H) = A$. □

Corollary 3.9. *If $A \in \wp^*(H)$, one has $D(H) \odot A = A \odot D(H) = C_\gamma(A)$.*

Corollary 3.10. *$D(H)$ is a γ-part of H.*

4. γ-COMPLETE HYPERGROUPS

We assume that $\phi_H : H \longrightarrow H/\gamma^*$ is the canonical projection.

Theorem 4.1. *Let H be a semi-hypergroup, the following conditions are equivalent*:

(i) $\forall (x, y) \in H^2, \forall \sigma \in \mathbb{S}_2, \forall a \in (x_{\sigma(1)} \circ y_{\sigma(2)}) : C_\gamma(a) = x_{\sigma(1)} \circ y_{\sigma(2)}$;
(ii) $\forall (x, y) \in H^2, \forall \sigma \in \mathbb{S}_2 : C_\gamma(x_{\sigma(1)} \circ y_{\sigma(2)}) = x_{\sigma(1)} \circ y_{\sigma(2)}$;
(iii) $\forall (m, n) \in \mathbb{N}^2, 2 \leq m, n, \forall (x_1, \ldots, x_n) \in H^n, \forall (y_1, \ldots, y_m) \in H^m, \forall (\sigma, \tau) \in \mathbb{S}_n \times \mathbb{S}_m$, *the following implication is valid*:

$$\prod_{i=1}^{n} x_{\sigma(i)} \cap \prod_{i=1}^{m} y_{\tau(i)} \neq \emptyset \Longrightarrow \prod_{i=1}^{n} x_{\sigma(i)} = \prod_{i=1}^{m} y_{\tau(i)}.$$

Proof. By Theorem 3.6, we have:

(i) $\Rightarrow$ (ii) $C_\gamma(x_{\sigma(1)} \circ y_{\sigma(2)}) = \bigcup_{a \in x_{\sigma(1)} \circ y_{\sigma(2)}} C_\gamma(a) = x_{\sigma(1)} \circ y_{\sigma(2)}$.

(ii) $\Rightarrow$ (i) From $a \in x_{\sigma(1)} \circ y_{\sigma(2)}$ follows $C_\gamma(a) \subseteq C_\gamma(x_{\sigma(1)} \circ y_{\sigma(2)}) = x_{\sigma(1)} \circ y_{\sigma(2)}$ since one has $C_\gamma(a) \cap (x_{\sigma(1)} \circ y_{\sigma(2)}) \neq \emptyset$, then $x_{\sigma(1)} \circ y_{\sigma(2)} \subseteq C_\gamma(a)$ and so $C_\gamma(a) = x_{\sigma(1)} \circ y_{\sigma(2)}$.

(ii) $\Rightarrow$ (iii) From $\prod_{i=1}^n x_{\sigma(i)} \cap \prod_{i=1}^m y_{\sigma(i)} \neq \emptyset$ follows that $a \in \prod_{i=1}^m y_{\tau(i)}$ exists such that $\prod_{i=1}^n x_{\sigma(i)} \subseteq y_{\tau(1)} \circ a \subseteq \prod_{i=1}^m y_{\tau(i)}$ and so $\prod_{i=1}^m y_{\tau(i)} = \prod_{i=1}^n x_{\sigma(i)}$.

(iii) $\Rightarrow$ (ii) From (iii) the following implication is valid: $\prod_{i=1}^n x_{\sigma(i)} \cap \prod_{i=1}^m y_{\tau(i)} \neq \emptyset \Longrightarrow \prod_{i=1}^n x_{\sigma(i)} = \prod_{i=1}^m y_{\tau(i)}$. Thus $\forall (a, b) \in H^2, \forall \sigma \in \mathbb{S}_2 : a_{\sigma(1)} \circ b_{\sigma(2)}$ is γ-part. □

Definition 4.2. A semi-hypergroup H is called *γ-complete* if it satisfies one of the equivalent condition of the preceding theorem.

Corollary 4.3. *Let H be a commutative hypergroup, then H is a γ-complete hypergroup if and only if H is a complete hypergroup.*

Theorem 4.4. *If H is a γ-complete semi-hypergroup and $\bar{x} \in H/\gamma^*$ is a member of quotient H/γ^* modulo a relation γ^*, then either there exist $(a, b) \in H^2$, $\sigma \in \mathbb{S}_2$ such that $\gamma^*(x) = a_{\sigma(1)} \circ b_{\sigma(2)}$ or $\gamma^*(x) = \{x\}$.*

Proof. It is enough to see that for Theorem 4.1, each product is a class of equivalence. □

Theorem 4.5. *If H is a γ-complete hypergroup, $\forall \bar{x} \in H/\gamma$ there exist $(a, b) \in H^2$, $\sigma \in \mathbb{S}_2$ such that $x = a_{\sigma(1)} \circ x_{\sigma(2)}$.*

Proof. It follows immediately from the preceding theorem and Theorem 4.1 in Freni (2002). □

Theorem 4.6. *Let H be a γ-complete hypergroup; then:*

(i) *$D(H)$ is the set of two-side identities of H;*
(ii) *H is regular and reversible.*

Proof. By Corollary 3.9 and Theorem 4.1 we have:

(i) If $u \in D(H)$, then $\forall a \in H$, $a \circ u = a \circ D(H) = C_\gamma(a)$;
(ii) Let $(a, e, a') \in H^3$, $\sigma \in \mathbb{S}_2$ be such that $e \in a'_{\sigma(1)} \circ a_{\sigma(2)}$. Then from reflexivity of $D(H)$ we have $a_{\sigma(1)} \circ a'_{\sigma(2)} = D(H)$. □

Definition 4.7. A hypergroup H is said *γ-flat* if for every subhypergroup K of H, one has $D(K) = D(H) \cap K$.

Theorem 4.8. *Every γ-complete hypergroup is γ-flat.*

Proof. Let H be a γ-complete hypergroup and K a sub-hypergroup of H, then we have $D(H) \cap K = \{e \in K \mid \forall x \in H, x \in e \circ x \cap x \circ e\} \subseteq D(K)$. Furthermore, if γ_K is the restriction of γ on K, then

$$y \in C_\gamma^K(x) \Longleftrightarrow y\gamma_K x \Longrightarrow y\gamma_H^* x \Longrightarrow y \in C_\gamma^H(x) \Longrightarrow C_\gamma^K(x) \subseteq C_\gamma^H(x).$$

It is clear that $D(H) \cap K \neq \emptyset$. If $x \in D(H) \cap K \subseteq D(K)$ one has $C_\gamma^K(x) = D(K)$, $C_\gamma^H(x) = D(H)$, and so $D(K) \subseteq D(H)$. Therefore, $D(K) \subseteq D(H) \cap K$. □

Theorem 4.9. *Let S be a sub semi-hypergroup of a γ-complete hypergroup H such that $S \cap D(H) \neq \emptyset$, then $D(H) \subseteq S$. If K is a subhypergroup of H, one has $D(K) = D(H)$.*

Proof. If $x \in S \cap D(H)$, then $x \circ x \subseteq S \cap D(H)$, but for $\sigma = 1$ one has $x \circ x = C_\gamma(x \circ x) = D(H)$, from this follows $D(H) \subseteq S \cap D(H)$ that is $D(H) \subseteq S$.

If K is a subhypergroup, since $D(H) \cap K \neq \emptyset$, one has $D(H) \subseteq K$ and then from the preceding theorem one deduces $D(H) = D(K)$. □

5. γ-CYCLIC HYPERGROUPS

Definition 5.1. A hypergroup H is said to be *γ-cyclic with a generator x* if H/γ is a cyclic group generated by $\phi_H(x)$.

Definition 5.2. An element x of hypergroup H is called *γ-periodical of γ-period n* if $x^n \subseteq D(H)$ and $n = \min\{k \in N \mid x^n \subseteq D(H)\}$. We write $P_\gamma(x) = n$.

Corollary 5.3. *Let H be a commutative hypergroup, then H is a cyclic hypergroup if and only if H is a γ-cyclic hypergroup.*

Theorem 5.4. *Let H be a hypergroup and $x \in H$ an element of γ-period n then:*

(i) $\forall (s, t) \in \{1, 2, \ldots, n\}^2$ *with $s \neq t$, one has* $C_\gamma(x^s) \cap C_\gamma(x^t) \neq \emptyset$;
(ii) $\forall m > n$ *there exists* $r \in \{1, 2, \ldots, n\}$ *such that* $C_\gamma(x^m) = C_\gamma(x^r)$.

Proof. (i) We suppose absurd $u \in C_\gamma(x^s) \cap C_\gamma(x^t)$, by Freni (2002, Theorem 3.2) and Theorem 3.6 we have $\phi(x^s) = C_\gamma(u) = C_\gamma(x^t)$, from which $\phi(x^s) = \phi(x^t)$ and therefore if $s > t$, $\phi(x^{s-t}) = \phi(x^s)\phi(x^t) = 1$, thus $x^{s-t} \in D(H)$ and $P_\gamma(x) \leq s - t$, against the hypothesis.

(ii) By a divided algorithm, let q, r be such that $m = qn + r$ with $0 \leq r < n$, if $0 < r$ one has $\phi(x^m) = \phi((x^n)^q)\phi(x^r) = \phi(x^r)$, therefore $C_\gamma(x^m) = C_\gamma(x^r)$. If $r = 0$, then $\phi(x^m) = \phi((x^n)^q) = 1$ from which $x^m \subseteq D(H)$ and for this reason $C_\gamma(x^m) = D(H) = C_\gamma(x^n)$. □

Theorem 5.5. *Let H be a hypergroup and $x \in H$ an element of γ-period n and let $\langle\{x\}\rangle$ be the smallest γ-complete subhypergroups that contain $\{x\}$ then $\langle\{x\}\rangle = \bigoplus_{r=1}^n C_\gamma(x^r)$, where $\oplus$ denotes a union of disjoint sets $C_\gamma(x^r)$ for $r = 1, \ldots, n$.*

Proof. It is enough to show that for every $y \in \langle\{x\}\rangle$ there exists $m \in N$ such that $y \in C_\gamma(x^m)$. Let $y \in \langle\{x\}\rangle = C_\gamma(G_m)$ for some G_m, then $t \in G_m$ exists such that $y \in C_\gamma(t)$, $t \in G_m$ implies that $t_1, \ldots, t_m$ and $\sigma \in S_m$ exist such that $x \in \prod_{i=1}^n t_i$ and $t \in \prod_{i=1}^n t_{\sigma(i)}$, therefore $\phi_H(t_1), \ldots, \phi_H(t_m) = \phi_H(y)$ since by Corollary 3.10 we have $G_m \subseteq D(H)$.

On the other hand, $\forall i, \phi_H(t_i) = \phi_H(x)$ or $\phi_H(t_i) = (\phi_H(x))^{-1}$, from this it follows that $h \in Z$ exists such that $\phi_H(y) = (\phi_H(x))^h$. If $h \in nZ, \phi_H(y) = 1$, then $y \in D(H) = x^n$. If $h \notin nZ$, then $q \in Z, 0 < r < n$ exist such that $h = nq + r, \phi_H(x^r) = \phi_H(x^{qn})\phi_H(x^r)$, from which $y \in C_\gamma(x^r)$, by Theorem 5.4, we have $\langle\{x\}\rangle \subseteq \bigoplus_{i=1}^n C_\gamma(x^r)$. The inverse inclusion is immediate. □

Theorem 5.6. *If H is a γ-cyclic and γ-complete then it is commutative.*

Proof. It follows immediately from Theorem 4.1. □

Theorem 5.7. *Let H be a γ-cyclic and γ-complete hypergroup s-generated from h and cycle(h) = r, then H is a join space.*

Proof. It follows from Corsini and Leoreanu (2003, Theorem 284) and Corollary 5.3. □

Definition 5.8. If H is a γ-cyclic semi-hypergroup we call the cyclicly of H the $max\{cycl(a) \mid a \in H\}$ and we denote it $cyclic(H)$.

Theorem 5.9. *If H is a cyclic semi-hypergroup with a generator h such that* $cycle(h) = 2$ *and* $cyclic(H) = m$, *then* $H = h^m = D(H)$.

Proof. It similarly follows from Corsini and Leoreanu (2003, Theorem 287). □

Theorem 5.10. *Let H be a γ-cyclic and γ-complete hypergroup s-generated from h and* $cycle(h) = r$, *then* $H = \bigoplus_{i=2}^r h^i$.

Proof. If $r = 2$ it follows from Theorem 5.9. Let $r > 2$, from Theorem 4.1 and Corsini and Leoreanu (2003, Theorem 285) one has $h^{r-1} = D(H)$, for this reason $P_\gamma(h)$ divides $r - 1$, if there where $P_\gamma(h) = s < r - 1$, then $h^s = D(H)$, from which $h \in h^s \circ h = h^{s+1}$ with $s + 1 < r$, which is absurd because $cycle(h) = r$. Thus $P_\gamma(h) = r - 1$. By Theorem 5.5 one has $H = \bigoplus_{i=1}^{r-1} C_\gamma(h^r)$, but $C_\gamma(h) = h^r$ and for every t: $1 < t \leq r - 1$, one has $C_\gamma(h^t) = h^t$. For this reason $H = \bigoplus_{i=2}^r h^i$. □

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